A remarkable representation of the Clifford group

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Outline

- Two useful groups in physics
- The Zak basis for finite systems
- An application to SIC-POVMs
Heisenberg Groups

Heisenberg groups can be defined in terms of upper triangular matrices

\[
\begin{pmatrix}
1 & x & \phi \\
0 & 1 & p \\
0 & 0 & 1 \\
\end{pmatrix}
\]

where \( x, p, \phi \) are elements of a ring, \( R \).

- \( R = \mathbb{R} \) - a three dimensional Lie group whose Lie algebra includes the position and momentum commutator
- \( R = \mathbb{Z}_N \) - a finite dimensional Heisenberg group \( H(N) \)
- \( R = \mathbb{F}_p^k \) - an alternative finite dimensional group
## Finite and CV systems

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The Clifford group

Write elements of $H(N)$ as

$$D_{ij} = \tau^{ij} X^i Z^j$$

where $\tau = -e^{i\pi/N}$, $X|j\rangle = |j+1\rangle$ and $Z|j\rangle = \omega^j |j\rangle$.

Then the composition law is

$$D_{ij}D_{kl} = \tau^{kj-il} D_{i+k,j+l}$$

The **Clifford group** is the normalizer of $H(N)$ i.e. all unitary operators $U$ such that

$$UD_{ij}U^\dagger = \tau^{k'} D_{i',j'}$$
The Heisenberg group $H(N)$ is defined by generators $\tau, X, Z$ with relations

$$ZX = \tau^2 XZ, \quad X^N = Z^N = 1$$

- There is a unique unitary representation [Weyl]
- The standard representation is to choose $Z$ to be diagonal.
Definition of the basis

The Heisenberg group $H(N)$ is defined by generators $\tau, X, Z$ with relations

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- There is a unique unitary representation [Weyl]
- The *standard* representation is to choose $Z$ to be diagonal.

But suppose $N = n^2$, then

$$Z^n X^n = X^n Z^n$$

- There is a (maximal) abelian subgroup $\langle Z^n, X^n \rangle$ of order $n^2 = N$
- So choose a basis in which this special subgroup is diagonal
The entire Clifford group is monomial in the new basis

Armchair argument:

- The Clifford group permutes the maximal abelian subgroups of $H(N)$
- It preserves the order of any element
- In dimension $N = n^2$ there is a *unique* maximal abelian subgroup where all of the group elements have order $\alpha n$
- Hence the Clifford group maps $\langle Z^n, X^n \rangle$ to itself
- The basis elements are permuted and multiplied by phases
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Armchair argument:

- The Clifford group permutes the maximal abelian subgroups of $H(N)$
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Theorem:
There exists a monomial representation of the Clifford group with $H(N)$ as a subgroup if and only if the dimension is a square, $N = n^2$. 
The new basis

In the Hilbert space $\mathcal{H}_N = \mathcal{H}_n \otimes \mathcal{H}_n$, the new basis is

$$X|r, s\rangle = \begin{cases} |r, s + 1\rangle & \text{if } s + 1 \neq 0 \mod n \\ \sigma^r |r, 0\rangle & \text{otherwise} \end{cases}$$

$$Z|r, s\rangle = \omega^s |r - 1, s\rangle$$

where the phases are $\omega = e^{2\pi i/N}$ and $\sigma = e^{2\pi i/n}$. Indeed,

$$X^n|r, s\rangle = \sigma^r |r, s\rangle \quad Z^n|r, s\rangle = \sigma^s |r, s\rangle$$

We have gone “half-way” to the Fourier basis

$$|r, s\rangle = \frac{1}{\sqrt{n}} \sum_{t=0}^{n-1} \omega^{-n tr} |nt + s\rangle .$$

i.e. apply $F_n \otimes \mathbb{I}$, where $F_n$ is the $n \times n$ Fourier matrix.
A SIC is a set of $N^2$ vectors $\{|\psi_i\rangle \in \mathbb{C}^N\}$ such that

$$|\langle \psi_i | \psi_j \rangle|^2 = \frac{1}{N+1} \quad \text{for } i \neq j$$

- A 2-design with the minimal number of elements.
- A special kind of (doable) measurement.
- Potentially a “standard quantum measurement” (cf. quantum Bayesianism)
Zauner’s Conjecture

SICs exist in every dimension

They can be chosen so that they form an orbit of $H(N)$ i.e. the SIC has the form $D_{ij}|\psi\rangle$ and there is a special order 3 element of the Clifford group such that

$$U_z|\psi\rangle = |\psi\rangle$$
Zauner’s Conjecture

SICs exist in every dimension

- Exact solutions in dimensions 2 – 16, 19, 24, 28, 35 and 48
- Numerical solutions in dimensions 2 – 67

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- All available evidence supports this conjecture
Images of SICs in the probability simplex

\[ |\psi\rangle = (\psi_00, \psi_{01}, \psi_{10}, \ldots)^T = (\sqrt{p_{00}}, \sqrt{p_{01}}e^{i\mu_{01}}, \sqrt{p_{10}}e^{i\mu_{10}}, \ldots)^T \]

↓ image w.r.t the basis

prob vector = \((p_{00}, p_{01}, p_{10}, \ldots)^T\)

Consider the equations

\[ \langle \psi | X^{nu} Z^{nv} | \psi \rangle = \sum_{r,s} p_{rs} \sigma^{ru+sv} \]
Images of SICs in the probability simplex

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Consider the equations

\[ |\langle \psi | X^{nu} Z^{nv} |\psi \rangle|^2 = \sum_{r,s} \sum_{r',s'} p_{rs} p_{r's'} \sigma^{(r-r')u+(s-s')v} \]
Images of SICs in the probability simplex

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Consider the equations

\[
\frac{1}{N + 1} = \sum_{r,s} \sum_{r',s'} p_{rs} p_{r's'} \sigma^{(r-r')u+(s-s')v}
\]
Images of SICs in the probability simplex

\[ |\psi\rangle = (\psi_{00}, \psi_{01}, \psi_{10}, \ldots)^T = (\sqrt{p_{00}}, \sqrt{p_{01}}e^{i\mu_{01}}, \sqrt{p_{10}}e^{i\mu_{10}}, \ldots)^T \]

\[ \downarrow \text{ image w.r.t the basis} \]

\[ \text{prob vector} = (p_{00}, p_{01}, p_{10}, \ldots)^T \]

Consider the equations

\[ \frac{1}{N+1} = \sum_{r,s} \sum_{r',s'} p_{rs} p_{r's'} \sigma^{(r-r')}u + (s-s')v \]

Take the Fourier transform

\[ \sum_{r,s} p_{rs}^2 = \frac{2}{N+1} \]

\[ \sum_{r,s} p_{rs} p_{r+x,s+y} = \frac{1}{N+1} \quad \text{for} \ (x, y) \neq (0, 0) \]
SICs are nicely aligned in the new basis

**Geometric interpretation:** When the SIC is projected to the basis simplex, we see a regular simplex centered at the origin with $N$ vertices.

The new basis is nicely orientated...
Solutions to the SIC problem

- Dimension $N = 2^2$ is now trivial
- Dimension $N = 3^2$ can be solved on a blackboard
- Dimension $N = 4^2$ can be solved with a computer
Solving the SIC problem in dimension $N = 2^2$

First write the SIC fiducial as

$$|\psi\rangle = \begin{pmatrix} \psi_{00} \\ \psi_{01} \\ \psi_{10} \\ \psi_{11} \end{pmatrix}$$

In the monomial basis, Zauner’s unitary is

$$U_z = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence, Zauner’s conjecture, $U_z|\psi\rangle = |\psi\rangle$, gives us

$$|\psi\rangle = \begin{pmatrix} a \\ a \\ a \\ be^{i\theta} \end{pmatrix}$$
Solving the SIC problem in dimension $N = 2^2$

The SIC fiducial is

$$|\psi\rangle = \begin{pmatrix} a \\ a \\ a \\ b e^{i\theta} \end{pmatrix}$$

Then we have two conditions,

\[
\begin{align*}
\text{Norm} = 1 & \Rightarrow 3a^2 + b^2 = 1 \quad (1) \\
\sum p_{rs}^2 = \frac{2}{N+1} & \Rightarrow 3a^4 + b^4 = \frac{2}{5} \quad (2)
\end{align*}
\]

Solving these equations gives

\[
\begin{align*}
a &= \sqrt{\frac{5 - \sqrt{5}}{20}} \\
b &= \sqrt{\frac{5 + 3\sqrt{5}}{20}}
\end{align*}
\]
Solving the SIC problem in dimension $N = 2^2$

Plugging these values into the equation

$$|\langle \psi | X | \psi \rangle|^2 = \frac{1}{5}$$

Gives

$$\frac{3 - \sqrt{5}}{20} + \frac{1 + \sqrt{5}}{10} \cos^2 \theta = \frac{1}{5}$$

Hence

$$\theta = \frac{(2\lambda + 1)\pi}{4} \quad \lambda = 0, 1, 2, 3$$

Conclusion: The Zauner eigenspace contains the fiducials

$$|\psi_\lambda\rangle = \sqrt{\frac{5 - \sqrt{5}}{20}} \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ \sqrt{2 + \sqrt{5}} e^{\pi i/4} i^\lambda \end{array} \right)$$
Solving the SIC problem in dimension $N = 3^2$

In the new basis, we can solve the SIC problem on the blackboard...
Solving the SIC problem in dimension $N = 3^2$

Zauner’s conjecture implies that

$$|\psi\rangle = -z_1 \omega^7 |1, 1\rangle - z_2 \omega |2, 2\rangle + z_3 (\omega^6 |0, 2\rangle + |1, 0\rangle + \omega^8 |2, 1\rangle) + z_4 (\omega^6 |0, 1\rangle + |2, 0\rangle + \omega^5 |1, 2\rangle).$$

$$z_1 = \sqrt{p_1} e^{i\mu_0} \quad z_2 = \sqrt{p_2} e^{-i\mu_0} \quad z_3 = \sqrt{p_3} e^{i\mu_3} \quad z_4 = \sqrt{p_4} e^{i\mu_4}.$$
Solving the SIC problem in dimension $N = 3^2$

The absolute values:

$$p_1 = a_1 + b_1 , \quad p_2 = a_1 - b_1 , \quad p_3 = a_3 + b_3 , \quad p_4 = a_3 - b_3$$

$$a_1 = \frac{1}{40} \left( 5 - s_05\sqrt{3} + s_03\sqrt{5} + \sqrt{15} \right)$$

$$b_1 = \frac{s_2}{60} \sqrt{15 \left( \sqrt{15} + s_0\sqrt{3} \right)}$$

$$a_3 = \frac{1}{120} \left( 15 + s_05\sqrt{3} - s_03\sqrt{5} - \sqrt{15} \right)$$

$$b_3 = \frac{s_1}{60} \sqrt{5 \left( -18 - s_07\sqrt{3} + s_06\sqrt{5} + 5\sqrt{15} \right)}$$

where $s_0 = s_1 = s_2 = \pm 1$
Solving the SIC problem in dimension $N = 3^2$

The phases:

$$e^{i\mu_0} = \frac{1}{2} + c_0 - is_1 \sqrt{\frac{1}{2} - c_0}$$

$$e^{i\mu_3} = q^{m_3} \left( -\sqrt{\frac{1}{2} - c_1 + c_2 + is_1 s_2 \sqrt{\frac{1}{2} + c_1 - c_2}} \right)^{\frac{1}{3}}$$

$$e^{i\mu_4} = q^{m_4} \left( -\sqrt{\frac{1}{2} - c_1 - c_2 + is_1 s_2 \sqrt{\frac{1}{2} + c_1 + c_2}} \right)^{\frac{1}{3}}$$

$$c_0 = \frac{1}{8} \sqrt{2(6 + s_0 \sqrt{3} - \sqrt{15})}$$

$$c_1 = \frac{s_0}{8} \sqrt{9 - s_0 4\sqrt{3} + s_0 3\sqrt{5} - 2\sqrt{15}}$$

$$c_2 = \frac{s_1 s_0}{24} \sqrt{15(-19 + s_0 12\sqrt{3} - s_0 9\sqrt{5} + 6\sqrt{15})}$$
Solving the SIC problem in dimension $N = 4^2$

The new basis allows us to solve the SIC problem in dimension 16...

The solutions are given in a number field

$$\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{13}, \sqrt{17}, r_2, r_3, t_1, t_2, t_3, t_4, \sqrt{-1}),$$

of degree 1024, where

$$r_2 = \sqrt{\sqrt{221} - 11} \quad r_3 = \sqrt{15 + \sqrt{17}}$$

$$t_1 = \sqrt{15 + (4 - \sqrt{17})r_3 - 3\sqrt{17}}$$

$$t_2^2 = ((3 - 5\sqrt{17})\sqrt{13} + (39\sqrt{17} - 65))r_3$$

$$\quad + ((16\sqrt{17} - 72)\sqrt{13} + 936))t_1 - 208\sqrt{13} + 2288$$

$$t_3 = \sqrt{2 - \sqrt{2}} \quad t_4 = \sqrt{2 + t_3}$$
Conclusion

- A basis were every element of the Clifford group is a monomial matrix
- The SICs are nicely orientated in the new basis
- The solutions to the SIC problem in dimensions 4, 9 and 16 are given entirely in terms of radicals, as expected (but not understood!)
- The result can be extended to non-square dimensions $kn^2$
- Are there other applications in quantum information?

$N = n^2$ : QIC vol 12, 0404 (2012), arXiv:1102.1268
$N = kn^2$ : in preparation